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Analytical modelling of dynamic plastic buckling of an axially loaded strain-rate sensitive bar

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Abstract

Dynamic plastic buckling of a bar is analytically studied based on some simplifying assumptions. The bar is simply supported and subjected to an axial step-force at its ends. The material is assumed to have linearly strain hardening and the strain-rate effect is considered by employing Malvern's over-stress model. Shanley's assumption of no unloading when plastic buckling occurs is adopted. A linear differential equation of motion of the bar is thus established. A stability condition is then derived by means of the method of amplifying function. Expressions of buckling half-wavelength and critical buckling load are obtained. The results are compared with those of a strain-rate insensitive bar. It is found that the strain-rate effect has a significant influence on the dynamic plastic buckling of bars.

1. Introduction

Buckling of structures has continuously drawn engineers' attention because of the increasing application of high strength, lower density materials in engineering. In some fields, such as transportation and aerospace engineering, structures and their members are often subjected to dynamics loads. Consequently, the well-developed classical theory of static buckling of structures is not applicable and the theory of dynamic buckling is put forward (Lindberg and Florence, 1987).

Dynamic buckling differs from static buckling of structures in several aspects. First, dynamic buckling is time-dependent, so that a key problem arising is how to determine the critical time, namely the time when buckling initiates. Accordingly, a buckling criterion is required not only to determine the buckling loads, as usually done in the classical analysis of static buckling, but also to deduce the buckling time.

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Secondly, stress waves appear in the structure immediately after the dynamic load (a pulse or impact) is applied. Evidence has shown that dynamic buckling usually occurs in the initial period of the stress wave propagation. Thus advanced studies on the dynamic buckling of structures should incorporate the effect of stress waves. Thirdly, for dynamic buckling in plastic range, i.e. dynamic plastic buckling, the strain-rate effect should be considered for the structures made of strain-rate sensitive materials. This aspect is the main concern of the present paper. To ease mathematical derivation, so as to highlight the physical significance of the unique features of dynamic plastic buckling, the simplest structural member, a straight bar, is examined in this paper.

In recent years, the strain-rate effect has been one focus of several studies of dynamic plastic behavior of structures (Jones, 1989; Tam and Calladine, 1991; Karagiozova and Jones, 1995a, b). The study of a structural model under impact revealed that, in some cases, the strain-rate effect is equally important to the inertia effect (Su et al., 1995).

A main difficulty in theoretical analysis and numerical computation, e.g. using finite element method, of dynamic plastic buckling of structures is the absence of a widely accepted criterion of stability. Theoretically strict stability criteria based on the Liapunov second method have been applied to dynamic elastic systems, namely conservative systems (Knops and Wikes, 1966; Jie, 1996), and limited cases of nonconservative systems (Leipholz, 1980; Jie, 1997). Stability conditions given by the Liapunov second method are usually sufficient but not necessary. On the other hand, the Liapunov second method has found little use in the study of dynamic plastic buckling of structures due to the extensive difficulty, or even impossibility, in forming the Liapunov functions. Lee (1977, 1978) proposed a concept of quasi-bifurcation in dynamics of elastoplastic continua and derived a buckling criterion for a bar under an axial plastic compressive wave. Wang and Ru (1985) established a stability criterion of energy and applied it to deduce a stability condition of a cylinder shell under axial impact. So far the most commonly used buckling criterion refers to the Method of Amplifying Function (MAF). In the MAF, the flexural shape of a dynamically deforming bar is supposed to be compounded by a series of harmonic modes, based on the mathematical concept of Fourier transformation. Each mode grows with time at its own rate. Among them there exists an optimum mode which grows fastest with time. The optimum mode should be taken as the buckling mode and its growth determines the buckling time. In their pioneer work on dynamic plastic buckling of a flying bar impacting on a rigid target, Abrahamson and Goodier (1966) successfully applied the MAF to derive the buckling mode and found good experimental agreement. Since the, the MAF has been broadly applied to analyze the dynamic buckling of various kinds of structural members (Lindberg and Florence, 1987), e.g. bars, plates and shells.

A common method in studying the dynamic elastoplastic buckling of a bar is numerical computations, e.g. via finite difference method (Havashi and Sano, 1972; Ari-Gur et al., 1982) and finite element method (Jie, 1991). Some factors, which are very difficult to be considered analytically, such as large deformation, unloading in the plastic stage, stress wave propagation and strain-rate effect, etc., can be included in the numerical computations. However, the computational methods have their own drawback of only showing the overall effects of all the considered factors rather than the separate effects. Besides, the convergence of computation is often difficult to achieve for a dynamic plastic problem. In contrast to numerical computations, simplified models have been found to be effective in describing the dynamic plastic buckling of bars; while these require relatively easy mathematical treatments, they provide profound physical descriptions and comprehensive results (Jones and de Reis, 1980; Gary, 1983; Karagiozova and Jones, 1992a, b, 1995a, b, 1996a, b).

In a preliminary work (Jie, 1995), Malvern's over-stress model (Malvern, 1950, 1951) was adopted to take into account the strain-rate effect in the analysis of dynamic plastic buckling of a perfectly elastoplastic bar. The results were then compared with those of existing theory and the experiment of 6061-T6 aluminum bars (Abrahamson and Goodier, 1996). It has been found that the inclusion of the strain-rate effect makes the theoretical predictions closer to the experimental ones. The present paper extends the study of Jie (1995) to a linearly strain hardening elastoplastic bar. The Malvern over-stress model is still employed to establish the dynamic stress–strain (σ–ε) relation of the bar. To model the problem analytically, the following basic assumptions are made in order to linearize the governing differential equation of the bar:

(i) The impact load is a step function of time.
(ii) Shanley's assumption (Shanley, 1947) for quasi-static plastic buckling of bars holds in the dynamic case, i.e. there is no unloading in the plastic stage when buckling initiates.
(iii) The effect of stress waves on dynamic buckling is neglected.
(iv) The elastic strain rate is ignored.

For the present configuration, the assumption (ii) has been affirmed by experiments (Jie et al., 1992).

Under the basic assumptions, a condition of stability, the buckling half-wavelength and the critical buckling load are deduced by means of the MAF. The results can retrograde to those of the strain-rate insensitive bar (Abrahamson and Goodier, 1966) as well as the rate-sensitive elastic, perfectly plastic bar (Jie, 1995), respectively. It is found that the strain-rate effect brings about some notable characteristics for dynamic plastic buckling of bars which have not been revealed before.

2. Governing equations

For linearly strain hardening elastoplastic materials, an approximation of Malvern's uniaxial dynamic relation is (Malvern, 1950, 1951)

$$E\dot{\varepsilon} = D\left(\sigma - \sigma_y - E_{\rm h}\varepsilon\right) + \dot{\sigma} \tag{1}$$

where $\dot{()} = \partial/\partial t()$, t is time, E Young's modulus, $E_{\rm h}$ the strain hardening modulus, σ_y the quasi-static yield stress. D is a material constant; for most metals, $D = 10^6\ {\rm s}^{-1}$ (Malvern, 1950, 1951).

Consider a bar in Fig. 1 with length l and two ends hinged. $y(x, t)$ denotes the flexural deflection with respect to the initial imperfection $y_0(x)$ of the bar. Hence, the total deflection of the bar is $y(x, t) + y_0(x)$. A pair of step force, expressed as $P(t) = 0$ when $t < 0$ and $P(t) = P$ when $t \geqslant 0$, is axially applied on the both ends of the bar. Here, P is constant. The dynamic equilibrium equation of the bar is

$$\rho A\ddot{y} = -M'' - P\left(y'' + y_0''\right) \tag{2}$$

where $()' = \partial/\partial x()$, ρ is the material density, A is the cross-sectional area of the bar and M is the bending moment. From eqn (1) and the assumptions (2) and (4), it is derived that

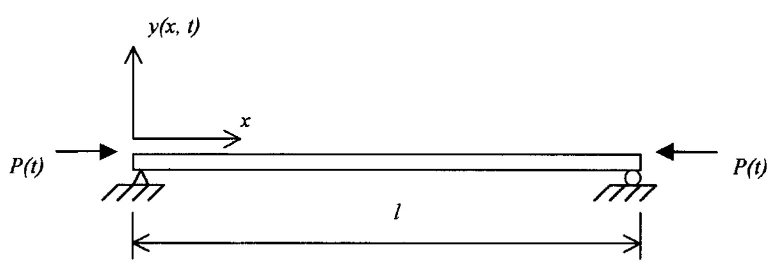

Fig. 1. A simply supported bar under axial dynamic loading.

$$M = \frac{EI\dot{y}''}{D} + E_h I(y + y_0)'' \tag{3}$$

in which I is the second moment of the cross-section.

Suppose $y = kw$, $y_0 = kw_0$, $x = k\xi$, $t = k\tau/c$, where $k = \sqrt{I/A}$ is radius of gyration of the cross-section and $c = \sqrt{E/\rho}$ is velocity of longitudinal elastic wave. The nondimensional dynamic equilibrium equation of the bar is then written as

$$\dot{w}'''' + \alpha\gamma w'''' + \gamma\mu w'' + \gamma\ddot{w} = -\gamma\mu w_0'' - \alpha\gamma w_0'''' \tag{4}$$

Now in eqn (4) and following equations, $\dot{()} = \partial/\partial\tau()$ and $()' = \partial/\partial\xi()$. $\alpha = E_h/E$ is the coefficient of strain hardening; $\gamma = kD/c$ is a nondimensional parameter representing the strain-rate effect and the velocity of longitudinal elastic wave in the bar; and $\mu = P/EA$ is the nondimensional dynamic load.

3. Conditions of stability and buckling

The basic idea of the MAF is observed in the following analysis. By using Fourier transformation $w(\xi,\tau) = \int_0^\infty g(\eta,\tau)\sin\xi\eta\,\mathrm{d}\eta$ and $w_0(\xi) = \int_0^\infty g_0(\eta)\sin\xi\eta\,\mathrm{d}\eta$, with η being the nondimensional wave number and $g(\eta,\tau)$ or initially g_0, being the amplitude of the corresponding harmonic mode, eqn (3) is recast as

$$\gamma\ddot{g} + \eta^4\dot{g} + \left(\alpha\gamma\eta^4 - \gamma\mu\eta^2\right)g = \left(\gamma\mu\eta^2 - \alpha\gamma\eta^4\right)g_0 \tag{5}$$

whilst the initial conditions are given by

$$g(\eta,0) = 0, \quad \dot{g}(\eta,0) = 0 \tag{6}$$

By definition, the amplifying function of the system, which relies on the wave number η and varies with time, is the ratio of the total amplitude of the harmonic mode at any time with its initial value, i.e.

$$A_m(\eta,\tau) = \frac{g + g_0}{g_0} \tag{7}$$

Let the identifying function of eqn (5) be $\Delta = \eta^8 + 4\gamma^2\eta^2(\mu - \alpha\eta^2)$. If $\Delta = 0$, it is solved from eqns (5) and (6) that

$$A_m(\eta,\tau) = \left(1 + \frac{\eta^4\tau}{2\gamma}\right)\exp\left(-\frac{\eta^4\tau}{2\gamma}\right) \tag{8}$$

It is evident that

$$\lim_{\tau\to\infty} A_m(\eta,\tau) = 0,$$

so that in this case buckling will not occur.

However, if $\Delta \neq 0$, then we obtain

$$A_m(\eta,\tau) = \frac{\gamma\left(\theta_2 e^{\theta_1\tau} - \theta_1 e^{\theta_2\tau}\right)}{\sqrt{\Delta}} \tag{9}$$

with

$$\theta_1 = \frac{\left(-\eta^4 + \sqrt{\Delta}\right)}{2\gamma} \quad \text{and} \quad \theta_2 = \frac{\left(-\eta^4 - \sqrt{\Delta}\right)}{2\gamma} \tag{10}$$

In this case only when

$$\mu - \alpha\eta^2 > 0, \quad \lim_{\tau\to\infty} A_m(\eta,\tau) = \infty$$

so buckling can occur.

Thus, a necessary (but not sufficient) condition of buckling is

$$\eta^2 < \frac{\mu}{\alpha} \tag{11}$$

Under condition (11), the dominant buckling mode should make $A_m(\eta,\tau)$ grow fastest with τ (or time t); in other words, this mode should make $\theta_1(\eta)$ take its maximum value.

Let

$$\frac{\mathrm{d}\theta_1(\eta)}{\mathrm{d}\eta} = 0,$$

it is derived that

$$\frac{2\mu\eta^6}{\gamma^2} = \left(2\alpha\eta^2 - \mu\right)^2 \tag{12}$$

Eqn (12) gives a formula to determine the buckling mode of a rate sensitive bar. For rate insensitive materials, $\gamma = \infty$, eqn (12) becomes

$$\eta^2 = \frac{\mu}{2\alpha} \tag{13}$$

Eqn (13) is in accordance with the previous result given by Abrahamson and Goodier (1966). For elastic, perfectly plastic materials, $\alpha = 0$ in eqn (12), so that

$$\eta^6 = \frac{\mu\gamma^2}{2} \tag{14}$$

Thus, a formula derived by Jie (1995) is regained.

Taking η^2 as variable, roots of eqn (12) can be expressed analytically (see Appendix A). There is only one real root for η^2, whilst two other roots are complex numbers, i.e. they include imaginary part. On the other hand, expression of the real root is too complicated to be applied practically. For these reasons, one (and only one) approximate solution of eqn (12) is required.

Ordinarily parameter η is far less than 1. To approximately solve eqn (12), initially its left side is set to be zero. Consequently, the first order recurrent solution is obtained, which is expressed in eqn (13). Then by substituting eqn (13) into the left side of eqn (12), the second order recurrent solution is obtained as

$$\eta^2 = \frac{\mu}{2\alpha} \pm \frac{\mu^2}{4\gamma\alpha^{5/2}} \tag{15}$$

It is easy to prove that in order to make θ_1 a maximum, the solution given by eqn (15) should take the negative sign, i.e.

$$\eta^2 = \frac{\mu}{2\alpha} - \frac{\mu^2}{4\gamma\alpha^{5/2}} \tag{16}$$

in which $\mu/2\gamma\alpha^{3/2} < 1$. It can be proved that generally the solution given by eqn (16) makes $\mathrm{d}^2\theta_1/\mathrm{d}\eta^2 < 0$ (see Appendix B).

By substituting eqn (16) into eqn (12) and rearranging the latter, the error caused by recurrence solution is calculated as

$$\mathrm{err} \equiv \frac{2\mu\eta^6}{\gamma^2} - \left(2\alpha\eta^2 - \mu\right)^2 \approx \frac{3\mu^5}{8\gamma^3\alpha^{9/2}} \tag{17}$$

Suppose Γ is the wave length of the buckling mode. The nondimensional form of Γ is $\lambda = \Gamma/k$. On the other hand, the nondimensional wave length has a certain relationship with the nondimensional wave number, i.e. $\lambda = 2\pi/\eta$. From eqn (16), nondimensional buckling half-wavelength is solved as:

$$\frac{\lambda}{2} = \frac{\pi}{\sqrt{\dfrac{\mu}{2\alpha} - \dfrac{\mu^2}{4\gamma\alpha^{5/2}}}} \tag{18}$$

The possible buckling mode in Fig. 1 should satisfy $\Gamma/2 \leqslant l$ or $\lambda/2 \leqslant \xi_l$, where $\xi_l = l/k$ denotes the nondimensional length of the bar. Consequently, a condition of stability is derived as

$$\frac{\mu}{\alpha} - \frac{\mu^2}{2\gamma\alpha^{5/2}} < \frac{2\pi^2}{\xi_l^2} \tag{19}$$

Suppose the nondimensional critical buckling load of the bar is $\mu_{\mathrm{cr}} \equiv P_{\mathrm{cr}}/EA$, then it should satisfy

$$\frac{\mu_{\mathrm{cr}}}{\alpha} - \frac{\mu_{\mathrm{cr}}^2}{2\gamma\alpha^{5/2}} = \frac{2\pi^2}{\xi_l^2} \tag{20}$$

4. Computation and discussion

Consider a bar made of steel with $E = 200$ GPa, $\alpha = 0.018$ and $D = 10^6\ \mathrm{s}^{-1}$. Dimension of its cross-section is supposed to be $10 \times 2.5\ \mathrm{mm}^2$. Comparisons between the computed results in the present paper and the previous results given by Abrahamson and Goodier (1966) for rate insensitive materials are shown in Figs 2 and 3, respectively.

Fig. 2 shows the variations of the buckling half-wavelength with respect to the step force applied, in the rate-sensitive and rate-insensitive cases. It is observed that the half-wavelength of a rate-sensitive bar (made of steel) is significantly different from that of a rate-insensitive bar. If the bar is rate-insensitive, it is well known that its buckling half-wavelength decreases monotonically with the increment of dynamic load. If the bar is rate-sensitive, there exists a characteristic load μ_0 (about 0.34×10^{-3} in the example), at which the half-wavelength renders its minimum value. For a dynamic load less than μ_0, the half-wavelength decreases with the increase of the load, similar to the case of a rate-insensitive bar. However, for a dynamic load greater than μ_0, the half-wavelength increases with the increase of the load, which seems beyond the existing understanding on the buckling mode of a bar. Because the half-wavelength of buckling mode cannot exceed the length of the bar, only parts of the curves below the line, $\lambda/2 = \xi_l$, are significant. In the case of rate-sensitive bar, the line, $\lambda/2 = \xi_l$, has two intersections with the $\lambda/2 \sim \mu$

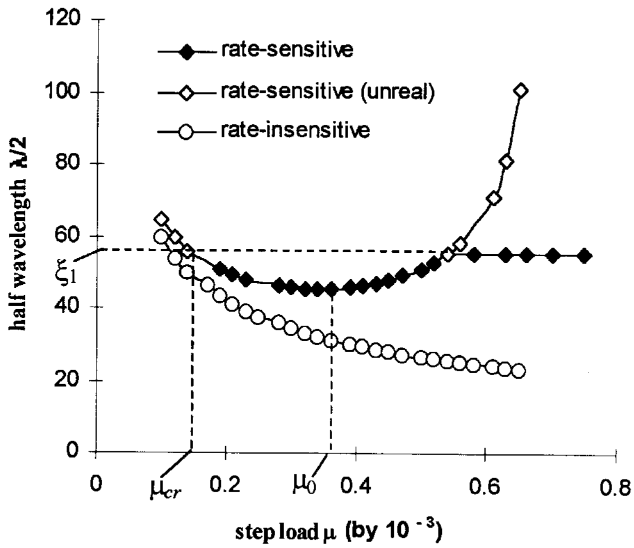

Fig. 2. Half-wavelength of buckling mode vs step load.

curve. The first intersection determines the critical buckling load μ_{cr}, whilst at the second intersection, the half-wavelength will remain to be equal to the length of the bar with the increase of the applied load. It is deduced from Fig. 2 that, for a rate-sensitive bar, two different applied loads may lead to identical buckling mode, which is a unique feature of dynamic plastic buckling in this case.

In Fig. 3, the critical buckling load μ_{cr} is plotted against length of the bar, ξ_l, separately for rate-sensitive and rate-insensitive bars. The $\mu_{cr} \sim \xi_l$ curve for the rate-sensitive bar has two branches, namely

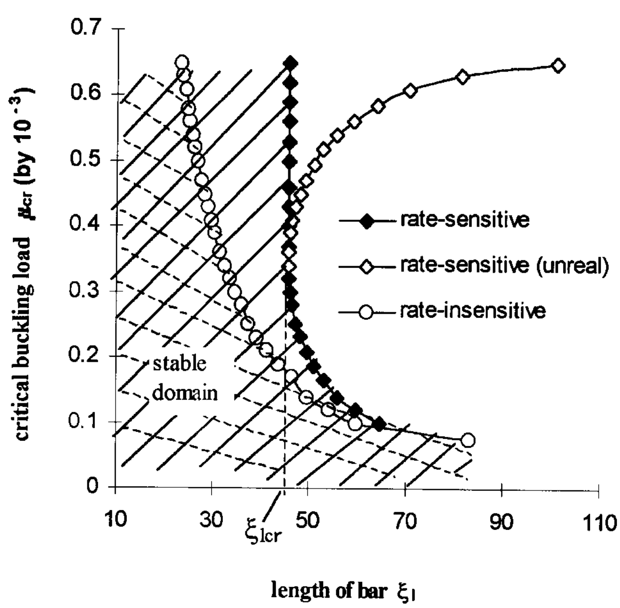

Fig. 3. Buckling load vs length of the bar.

the upper branch and the lower branch. Obviously, the upper branch is unrealistic for the critical buckling load. In Fig. 3, it is observed that when the bar is long, the critical buckling load of a rate-sensitive bar is rather close to that of a rate-insensitive bar. However, for a relatively short bar, a greater difference exists between the two cases, and the difference increases with the reduction in the length of the bar. That is to say, the strain-rate exerts a minor effect on the dynamic plastic buckling of a long bar but an essential effect on that of a rate-insensitive bar. It is also observed that the stable domain of a rate-sensitive bar is broader than that of a rate-insensitive bar. It is found that there exists a critical length ξ_{lcr} (about 45.5 in the example) for the rate-sensitive bar. If the length of the bar is less than ξ_{lcr}, the stable domain becomes infinite; in other words, there is no critical buckling load for the bar. Consequently, the buckling will never occur. This is a rather astonishing result that has never been explored before. In common knowledge, there always exists a critical buckling load for a bar, if the possible fracture of the bar is disregarded. Therefore, the above result needs a careful experimental verification. A reasonable explanation to this theoretically predicted phenomenon is that a short bar will become too strong and hard to lose its stability due to the strain-rate effect. On the other hand, for a bar with length greater than ξ_{lcr}, the buckling load decreases with the increase of the length, similar to the case of a rate-insensitive bar.

From eqn (17), it is estimated for the above example that the error caused by recurence solution is in the order of 10^{-6}, which is obviously a small quantity. Therefore, the recurence solution given by eqn (16) is sufficiently precise.

In general, it is not difficult to derive that, characteristic load is

$$\mu_0 = \alpha^{3/2}\gamma \tag{21}$$

Besides, the critical length of the bar, for which the critical buckling load approaches infinite, is

$$\xi_{lcr} = 2\pi\gamma^{-1/2}\alpha^{-1/4} \tag{22}$$

Discussion is necessary in regard to the dynamic elastoplastic σ–ε relation employed in the analysis. It should be pointed out that the Malvern's over-stress model is actually unrealistic for rate-sensitive metals, e.g. structural steel. The model somewhat exaggerates the increase of stress with strain rate. A commonly used σ–ε relation for dynamic plastic analysis is the Cowper–Symonds model, which is a power law formula (for details see section 1.3, Stronge and Yu, 1993). The Malvern's model is adopted in this paper mainly because it can lead to analytical results with not much loss of the physical characteristics of the problem. In fact, a simpler linear constitutive relation which is thought to be backed up by actual data, was used by Lindberg and Florence (1987) in the analysis of dynamic plastic buckling of plates and shells.

In this study, wave propagation in the bar is neglected. Therefore, a limit of end-shortening speed must exist under which the present solution is appropriate. The limit speed depends on the buckling time. For the wave propagation to be negligible, the buckling time must be greater than that for plastic longitudinal wave front traveling through the whole bar. However, as has been stated, a widely accepted buckling criterion is desired to determine the buckling time. This task may be left for later investigations.

Finally, it is easy to find out that all the results in this paper are appropriate to dynamic buckling of a viscoelastic bar of Maxwell type.

5. Conclusions

Dynamic plastic buckling of a bar under axial dynamic load has been analytically studied with the strain-rate effect being considered, and the following conclusions can be drawn.

(i) Expressions of the buckling half-wavelength and the critical buckling load are derived for a rate-sensitive bar made of Malvern material.
(ii) There exists a characteristic dynamic load in the load vs buckling half-wavelength curve of a rate-sensitive bar, at which the buckling half-wavelength takes its minimum value. When the applied load is smaller than the characteristic load, the buckling half-wavelength decreases with the increase of the load, whilst when it is larger than the characteristic load, the buckling half-wavelength increases with the increase of the load. Two different dynamic loads may cause the same buckling mode.
(iii) The stable domain of a bar under axial load is broadened by the rate sensitivity of its material.
(iv) Existence of a critical length is revealed for a rate-sensitive bar. For a bar shorter than the critical length, buckling will never occur.
(v) The strain-rate has less effect on the critical buckling load of a long bar.
(vi) General expressions of the characteristic load and the critical length are presented.

Appendix A. Solution of eqn (12)

In eqn (12), let $z = \eta^2$, there is

$$z^3 - \frac{2\alpha^2\gamma^2}{\mu}z^2 + 2\alpha\gamma^2 z - \frac{\mu\gamma^2}{2} = 0 \tag{A1}$$

The solution of eqn (A1) is obtained by employing the software MATHEMATICA. Three roots are, respectively,

$$z_1 = f_1 - \frac{f_2}{6\sqrt[3]{2}\mu f_3} + \frac{f_3}{3\sqrt[3]{4}\mu}$$

$$z_2 = f_1 + \frac{(1 + i\sqrt{3})f_2}{12\sqrt[3]{2}\mu f_3} - \frac{(1 - i\sqrt{3})f_3}{6\sqrt[3]{4}\mu}$$

$$z_3 = f_1 + \frac{(1 + i\sqrt{3})f_2}{12\sqrt[3]{2}\mu f_3} - \frac{(1 - i\sqrt{3})f_3}{6\sqrt[3]{4}\mu} \tag{A2}$$

where

$$f_1 = \frac{2\alpha^2\gamma^2}{3\mu}$$

$$f_2 = 24\alpha\mu^2\gamma^2 - 16\alpha^4\gamma^4$$

$$f_3 = \sqrt[3]{27\mu^4\gamma^2 - 72\alpha^3\mu^2\gamma^4 + 32\alpha^6\gamma^6 + 3\sqrt{3}\sqrt{27\mu^8\gamma^4 - 16\alpha^3\mu^6\gamma^6}3} \tag{A3}$$

Appendix B. Extreme value problem of θ_1

In eqn (10), let $z = \eta^2$, there is

$$\theta_1 = \frac{-z^2 + \sqrt{z^4 + 4\gamma^4 z(\mu - \alpha z)}}{2\gamma} \tag{B1}$$

It is proved that

$$\left.\frac{d\theta_1}{dz}\right|_{z=\frac{\mu}{2\alpha}-\frac{\mu^2}{4\alpha^2\sqrt{\alpha\gamma}}} \approx 0 \tag{B2}$$

The second order derivative of θ_1 is

$$\frac{d^2\theta_1}{dz^2} = -\frac{1}{\gamma}\left\{1 + \frac{2\left(z^3 - 2\alpha\gamma^2 z + \mu\gamma^2\right)^2}{\left[z^4 + 4\gamma^2 z(\mu - \alpha z)\right]^{3/2}} + \frac{2\alpha\gamma^2 - 3z^2}{\left[z^4 + 4\gamma^2 z(\mu - \alpha z)\right]^{1/2}}\right\} \tag{B3}$$

Numerical calculation shows that, when the parameters α, γ and μ take those values with physical significance, the η^2 (or z) given by eqn (16) makes the right side of eqn (B3) less than zero, which means at this point θ_1 takes its maximum value.

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